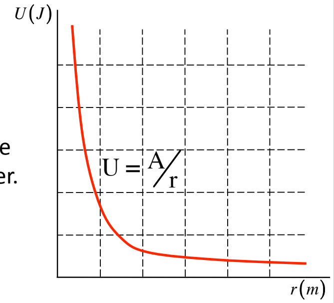
Problem 7.47

Determine the force associated with the potential energy function $U(r) = A/r$.

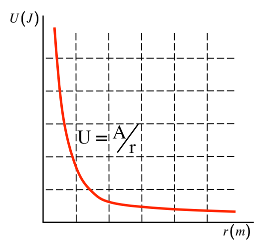


Potential energy functions and their associated force fields have an interesting relationship to one another. Specifically, the *rate* at which the potential energy CHANGES at a point is proportional to the force at that point. If you think about this conceptually, this makes perfect sense. If a force is really big in a region, you would expect an object in that region to change its kinetic energy fast. That would suggest that in the region, a small change in position should engender a big change in potential energy (remember, for a single force system, $W_{net} = \Delta KE = -\Delta U$).

Put a little differently, it kind of makes sense that if $\Delta U = -\int \vec{F} \bullet d\vec{r}$, it is true that $dU(r) = -\vec{F} \bullet d\vec{r}$. And if that is true, then:

$$|\vec{F}| = -\frac{dU(r)}{dr}.$$

Things only get slightly tricky when you note that you are relating a *scalar* field (the *potential energy* field) to a *vector* field (the *force* field), and additionally that you might have more than one dimension involved. That has all been accommodated in a very clever way. If you were to define the notation:



$$\frac{\partial}{\partial x}\hat{i}$$

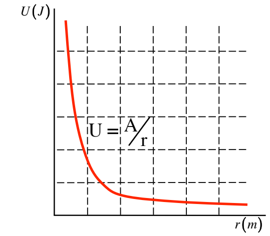
as "a derivative with respect to "x" holding all other variable constant (this is called a *partial derivative*), times a *unit vector* in the x-direction," then we could express our multi-variable force vector as:

$$\vec{F}(x,y,z) = -\left(\frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}\right)$$

That is, the x, y and z components of the force would equal, respectively, the *rate of change* of the potential energy function in the x-direction, times $\hat{i}$, plus the *rate of change* of the potential energy function in the y-direction, times $\hat{j}$, etc.

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)$$

In an attempt to abbreviate this, a *del operator* is defined as:



$$\vec{\nabla} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)$$

So the force can be written in its must succinct form as:

$$\vec{F} = -\vec{\nabla}U$$

The only additional twist to all of this is if the potential function is given in polar-spherical notation, like the problem we are trying to do. In that case

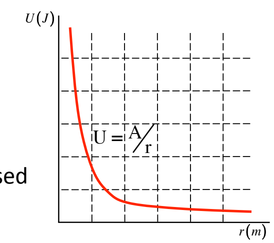
$$\vec{\nabla} = \left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\phi}\right)$$

(Yousa!) Fortunately, you won't need to know this for the AP test, but you can still use it here as:

$$\vec{F} = -\vec{\nabla}(Ar^{-1}) = -\left(\frac{\partial(Ar^{-1})}{\partial r}\hat{r} + \frac{1}{r}\cancelto{0}{\frac{\partial(Ar^{-1})}{\partial \theta}}\hat{\theta} + \frac{1}{r\sin\theta}\cancelto{0}{\frac{\partial(Ar^{-1})}{\partial \phi}}\hat{\phi}\right)$$

$$= \left(-(-1)Ar^{-2}\right)\hat{r} = \left(\frac{A}{r^2}\right)\hat{r}$$

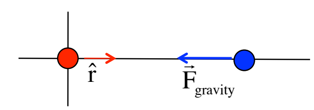
A couple of things to notice beyond the facts that *minus the slope* of the potential energy function shown to the right is, indeed, positive, and the force drops off as $1/r^2$.



Specifically, Newton's general gravitational force (an attraction between the bodies feeling the effect), expressed in polar-spherical coordinates, is

$$F = \frac{Gm_1m_2}{r^2}(-\hat{r})$$

The negative sign in this case makes sense if you think about what it suggests. Look at the sketch to the right. With the *radial unit vector* as defined, the attractive gravitational force on the blue body is in the $-\hat{r}$, just as the force function suggests. And if we derived the potential function for this force, we would get:



$$U = -\frac{Gm_1m_2}{r}$$

The point is that because this problems (i.e., Problem 7.47) has a given potential energy function that is *positive*, and our derived force also positive, it can be concluded that that force was REPULSIVE in nature.